Non-autonomous dynamics of wave equations with nonlinear damping and critical nonlinearity

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Abstract

The authors consider non-autonomous dynamical behavior of wave-type evolutionary equations with nonlinear damping and critical nonlinearity. These type of waves equations are formulated as non-autonomous dynamical systems (namely, cocycles). A sufficient and necessary condition for the existence of pullback attractors is established for norm-to-weak continuous non-autonomous dynamical systems, in terms of pullback asymptotic compactness or pullback $\kappa-$contraction criteria. A technical method for verifying pullback asymptotic compactness, via contractive functions, is devised. These results are then applied to the wave-type evolutionary equations with nonlinear damping and critical nonlinearity, to obtain the existence of pullback attractors. The required pullback asymptotic compactness for the existence of pullback attractors is fulfilled by some new a priori estimates for concrete wave type equations arising from applications. Moreover, the pullback $\kappa-$contraction criterion for the existence of pullback attractors is of independent interest.

Keywords: Non-autonomous dynamical systems; Cocycles; Wave equations; Nonlinear damping; Critical exponent; Pullback attractor.

Dedicated to Philip Holmes on the occasion of his 60th birthday

1 Introduction

Nonlinear wave phenomena occur in various systems in physics, engineering, biology and geosciences [4, 14, 21, 40, 31, 34]. At the macroscopic level, wave phenomena may be modeled by hyperbolic wave type partial differential equations. We consider the following non-autonomous wave equations with nonlinear damping, on a bounded domain Ω in $\mathbb{R}^3$, with smooth boundary $\partial\Omega$:

$$u_{tt} + h(u_t) - \Delta u + f(u,t) = g(x,t) \quad x \in \Omega \tag{1.1}$$

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subject to the boundary condition

$$u|_{\partial\Omega} = 0, \tag{1.2}$$

and the initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x). \tag{1.3}$$

Here h is the nonlinear damping function, f is the nonlinearity, g is a given external time-dependent forcing, and $\Delta = \partial_{x_1x_1} + \partial_{x_2x_2} + \partial_{x_3x_3}$ is the Laplace operator.

Equation (1.1) arises as an evolutionary mathematical model in various systems. For example, (i) modeling a continuous Josephson junction with specific h, g and f [27]; (ii) modeling a hybrid system of nonlinear waves and nerve conduct; and (iii) when $h(u_t) = ku_t$ and $f(u) = |u|^r u$, the equation (1.1) models a phenomenon in quantum mechanics [10, 15, 20, 40].

For the autonomous case of (1.1), i.e., when f and g do not depend on time t explicitly, the asymptotic behaviors of the solutions have been studied extensively in the framework of global attractors; see, for example, [1, 3, 4, 14, 21] for the linear damping case, and [16, 17, 18, 19, 38] for the nonlinear damping case.

In this paper, we consider the non-autonomous case, especially when the damping h is nonlinear and when the nonlinearity f has critical exponent (see below). For a non-autonomous dynamical system like (1.1)-(1.3), the solution map does not define a semigroup and instead, it defines a two-parameter *process*, or *cocycle*. Pullback attractors are appropriate geometric objects for describing asymptotic dynamics for cocycles. We will briefly introduce basic concepts for non-autonomous dynamical systems in §3. We will discuss the asymptotic dynamics of (1.1)-(1.3) via pullback attractors of the corresponding cocycle. This dynamical framework allows us to handle more general non-autonomous time-dependency; for example, the external force g needs to be neither almost periodic nor translation compact in time.

Our basic assumptions about nonlinear damping h, nonlinearity f and forcing g are as follows. Let $g(x,t)$ be in the space $L^2_{loc}(\mathbb{R}; L^2(\Omega))$, of locally square-integrable functions, and assume that the functions h and f satisfy the following conditions:

$$h \in \mathcal{C}^1(\mathbb{R}), \quad h(0) = 0, \quad h \text{ strictly increasing}, \tag{1.4}$$

$$\liminf_{|s|\to\infty} h'(s) > 0, \tag{1.5}$$

$$|h(s)| \leq C_1(1 + |s|^p), \tag{1.6}$$

where $p \in [1, 5)$ which will be given precisely later; $f \in \mathcal{C}^1(\mathbb{R}\times\mathbb{R}; \mathbb{R})$ and satisfies

$$F_s(v,s) \leqslant \delta^2 F(v,s) + C_\delta, \quad F(v,s) \geqslant -mv^2 - C_m, \tag{1.7}$$

$$|f_v(v,s)| \leq C_2(1 + |v|^q), \ |f_s(v,s)| \leq C_3(1 + |v|^{q+1}), \tag{1.8}$$

$$f(v,s)v - C_4 F(v,s) + mv^2 \geqslant -C_m, \quad \forall\ (v,s) \in \mathbb{R}\times\mathbb{R}, \tag{1.9}$$

where $0 \leqslant q \leqslant 2$, $F(v,s) = \int_0^v f(w,s)dw$ and δ, m are sufficiently small which will be determined in *Lemma* 5.3. The number $q = 2$ is called the *critical exponent*, since the nonlinearity f is not compact in this case (i.e., for a bounded subset $B \subset H_0^1(\Omega)$, in general, $f(B)$ is not precompact in $L^2(\Omega)$). This is an essential difficulty in studying the asymptotic

behavior even for the autonomous cases [1, 3, 4, 16, 17, 18, 19, 38]. The assumptions (1.4)-(1.6) on h are similar to those in [18, 19, 25, 38] for the autonomous cases, while the assumption $1 \leq p < 5$ is due to the need for estimating $\int_\Omega g(u_t)u$ by $\int_\Omega g(u_t)u_t$ and $\int_\Omega |\nabla u|^2$ via Sobolev embedding. Finally, the assumptions (1.7)-(1.9) are similar to the conditions used in Chepyzhov & Vishik [14] for non-autonomous cases but with linear damping.

Let us recall some recent relevant research in this area.

The existence of pullback attractors are established for the strongly dissipative non-autonomous dynamical systems such as those generated by parabolic type partial differential equations, e.g., the non-autonomous 2D Navier-Stokes equation and some non-autonomous reaction diffusion equations; see [2, 7, 8, 9, 11, 12, 13, 26] and the references therein. However, the situation for the hyperbolic wave type systems is less clear. For the linear damping case $h(v) = kv$ with a constant $k > 0$ and $q < 2$ (subcritical), Chepyzhov & Vishik [14] have obtained the existence of a uniform absorbing set when g is translation bounded in time (i.e., $g \in L_b^2(\mathbb{R}; L^2(\Omega)))$, and the existence of a uniform attractor when g is translation compact in time (i.e., $g \in L_c^2(\mathbb{R}; L^2(\Omega)))$.

Under the assumptions that g and $\partial_t g$ are both in the space of bounded continuous functions $\mathcal{C}_b(\mathbb{R}, L^2(\Omega))$, h has bounded positive derivative, and furthermore, f is of critical growth (i.e., $q = 2$), Zhou & Wang [42] have proved the existence of kernel sections and obtained uniform bounds of the Hausdorff dimension of the kernel sections. Caraballo et al. [6] have discussed the pullback attractors for the cases of linear damping and subcritical nonlinearity ($q < 2$) .

As in the autonomous case, some kind of compactness of the cocycle is a key ingredient for the existence of pullback attractors of cocycles. The corresponding compactness assumption in Cheban [11] is that the cocycle has a compact attracting set. Recently, Caraballo et al [7] have established a criterion for the existence of pullback attractors via pullback asymptotic compactness, and illustrated their results with the 2D Navier-Stokes equation.

For the autonomous *linearly* damped wave equations, Ball [4] proposed a method to verify the asymptotic compactness for the corresponding solution semigroup. This so-called energy method has been generalized [30, 32] to some non-autonomous cases. However, for our problem, due to the nonlinear damping, it appears difficult to apply the method of Ball [4]. Moreover, a decomposition technique [1, 14, 19, 21, 33, 36] has been successfully applied to verify the asymptotic smoothness of the corresponding solution semigroup for autonomous wave equations.

In this paper, after some preliminaries, we first introduce the *pullback $\kappa-$contraction* concept, a generalization of $\kappa-$contraction from autonomous systems to non-autonomous systems. Then we establish a criterion for the existence of pullback attractors, in terms of pullback $\kappa-$contraction or pullback asymptotic compactness. This criterion is for a class of "weakly" continuous cocycles (i.e., the so-called norm-to-weak continuous cocycles; see §3 below). Thirdly, we show that the pullback $\kappa-$contraction is not equivalent to the pullback asymptotic compactness, unless the cocycle mapping has a nested bounded pullback absorbing set (see $Definition$ 3.7 below). This fact is different from the autonomous semigroup cases. Moreover, we propose a technique for verifying pullback asymptotic compactness. Finally, we apply these results to show the existence of pullback attractors for the non-autonomous hyperbolic wave system (1.1)-(1.3).

Due to the difference between the cases $p = 1$ and $1 < p < 5$ for the nonlinear damping exponent p, we propose the following two kinds of assumptions.

Assumption I.

h satisfies (1.4)-(1.6) with $p = 1$ and there is a C_0 such that $C_0|u - v|^2 \leqslant (h(u) - h(v))(u - v)$;

g satisfies

$$\int_{-\infty}^{t} e^{\beta s} \int_{\Omega} |g(x,s)|^2 dxds < \infty \quad \text{for each} \quad t \in \mathbb{R}, \tag{1.10}$$

where $\beta(< C_0)$ is constant depending on the coefficients of h and f, which will be determined in the proof of *Lemma* 5.3;

f satisfies (1.7)-(1.9).

Assumption II.

h satisfies (1.4)-(1.6) with $1 \leqslant p < 5$; And

$$g \in L^{\infty}(\mathbb{R}, L^2(\Omega)); \tag{1.11}$$

In addition to (1.7)-(1.9), f satisfies also

$$F_s(v,s) \leqslant 0 \quad \text{for all} \quad (v,s) \in \mathbb{R} \times \mathbb{R}. \tag{1.12}$$

We remark that the technical hypotheses (1.11) and (1.12) in *Assumptions II* are mainly for the existence of pullback absorbing set; see *Lemma* 5.3 below or Haraux [23] for more details. Our method for verifying the asymptotic compactness allows us take some more general assumptions than (1.11)-(1.12).

For convenience, hereafter let $|\cdot|_p$ be the norm of $L^p(\Omega)$ $(1 \leqslant p < \infty)$, and C a general positive constant, which may be different in different estimates.

This paper is organized as follows. We present some background materials in §2, then prove a criterion on existence of pullback attractors in §3, and a technical method for verifying pullback asymptotic compactness is presented in §4. Finally, in §5, these abstract results are applied to a non-autonomous wave equation with nonlinear damping and critical nonlinearity, to obtain the existence of pullback attractors. We conclude the paper with some remarks in §6.

2 Preliminaries

2.1 Kuratowski measure of non-compactness

We briefly review the basic concept about the Kuratowski measure of non-compactness and recall its basic properties, which will be used to establish a criterion for the existence of pullback attractors.

Definition 2.1. *([21, 36]) Let X be a complete metric space and A be a bounded subset of X. The Kuratowski measure of non-compactness $\kappa(A)$ of A is defined as*

$$\kappa(A) = \inf\{\delta > 0 |\ A \text{ has a finite open cover of sets of diameter} < \delta\}.$$

If A is a nonempty, unbounded set in X, then we define $\kappa(A) = \infty$.

The properties of $\kappa(A)$, which we will use in this paper, are given in the following lemmas:

Lemma 2.2. *([21, 36]) The Kuratowski measure of non-compactness $\kappa(A)$ on a complete metric space X satisfies the following properties:*

(1) $\kappa(A) = 0$ *if and only if* $\bar{A}$ *is compact, where* $\bar{A}$ *is the closure of* A*;*

(2) $\kappa(\bar{A}) = \kappa(A)$, $\kappa(A \cup B) = \max\{\kappa(A), \kappa(B)\}$*;*

(3) If $A \subset B$, then $\kappa(A) \leqslant \kappa(B)$;

(4) If A_t is a family of nonempty, closed, bounded sets defined for $t > r$ that satisfy $A_t \subset A_s$, whenever $s \leq t$, and $\kappa(A_t) \to 0$, as $t \to \infty$, then $\bigcap_{t>r} A_t$ is a nonempty, compact set in X.

If in addition, X is a Banach space, then the following estimate is valid:

(5) $\kappa(A+B) \leqslant \kappa(A) + \kappa(B)$ for any bounded sets A, B in X.

2.2 Some useful properties for nonlinear damping function

In the following, we will recall some simple properties of the nonlinear damping function h, which will be used later.

Lemma 2.3. *([19, 25]) Let h satisfy* (1.4) *and* (1.5)*. Then for any $\delta > 0$, there exists a constant C_δ depending on δ such that*

$$|u-v|^2 \leqslant \delta + C_\delta (h(u) - h(v))(u-v) \quad \text{for all } u, v \in \mathbb{R}.$$

Moreover, condition (1.6) implies that

$$|h(s)|^{\frac{1}{p}} \leqslant C(1+|s|).$$

Therefore, we have

$$|h(s)|^{\frac{p+1}{p}} = |h(s)|^{\frac{1}{p}} \cdot |h(s)| \leqslant C(1+|s|)|h(s)| \leqslant C|h(s)| + Ch(s) \cdot s.$$

Combining this estimate with the Young's inequality and (1.4), we further obtain that

$$|h(s)|^{\frac{p+1}{p}} \leqslant C(1 + h(s) \cdot s) \quad \text{for all } s \in \mathbb{R}, \tag{2.1}$$

where the constant C is independent of s.

3 Criterion for the existence of pullback attractors

In this section, we first recall a few basic concepts for non-autonomous dynamical systems, including pullback $\kappa-$contraction, pullback asymptotic compactness and pullback attractor. Then we present criteria for existence of pullback attractors, in terms of $\kappa-$contraction or pullback asymptotic compactness.

Let X be a complete metric space, which is the state space for a non-autonomous dynamical system (NDS). As in [5, 11, 13], we define a non-autonomous dynamical system in terms of a cocycle mapping φ: $\mathbb{R}^+ \times \Sigma \times X \to X$ which is driven by an *autonomous* dynamical system θ acting on a parameter space Σ. In details, $\theta = \{\theta_t\}_{t\in\mathbb{R}}$ is a autonomous dynamical system on Σ, i.e., a group of homeomorphisms under composition on Σ with the properties

(i) $\theta_0(\sigma) = \sigma$ for all $\sigma \in \Sigma$;

(ii) $\theta_{t+\tau}(\sigma) = \theta_t(\theta_\tau(\sigma))$ for all $t, \tau \in \mathbb{R}$.

The cocycle mapping φ satisfies

(i) $\varphi(0,\sigma;x)=x$ for all $(\sigma,\,x)\in\Sigma\times X$;

(ii) $\varphi(s+t,\,\sigma;x)=\varphi(s,\theta_t(\sigma);\varphi(t,\sigma;x))$ for all $s,t\in\mathbb{R}^+$ and all $(\sigma,x)\in\Sigma\times X$.

Sometimes we say φ is a cocycle with respect to (w.r.t.) θ and denote this by (φ,θ). If, in addition, the mapping $\varphi(t,\sigma;\cdot):\ X\to X$ is continuous for each $\sigma\in\Sigma$ and $t\geqslant 0$, then we call φ is a *continuous cocycle*. If the mapping $\varphi(t,\sigma;\cdot):\ X\to X$ is *norm-to-weak continuous* for each $\sigma\in\Sigma$ and $t\geqslant 0$, that is, for each $\sigma\in\Sigma$ and $t\geqslant 0$, norm convergence $x_n\to x$ in X implies weak convergence $\varphi(t,\sigma;x_n)\rightharpoonup\varphi(t,\sigma;x)$, then we call φ is a *norm-to-weak continuous cocycle*. A continuous cocycle is obviously also a norm-to-weak continuous cocycle.

For convenience, hereafter, we will use the following notations:

$$\mathcal{B}\triangleq\{B\mid B\text{ is bounded in }X\};\quad \varphi(t,\,\sigma;\,B)\triangleq\{\varphi(t,\,\sigma;\,x_0)\mid x_0\in B\}.$$

Definition 3.1. *([11]) A family of bounded sets $\mathscr{B}=\{B_\sigma\}_{\sigma\in\Sigma}$ of X is called a bounded pullback absorbing set for the cocycle φ with respect to (w.r.t.) θ, if for any $\sigma\in\Sigma$ and any $B\in\mathcal{B}$, there exists $T=T(\sigma,B)\geqslant 0$ such that*

$$\varphi(t,\,\theta_{-t}(\sigma);B)\subset B_\sigma\quad\textit{for all } t\geqslant T.$$

Definition 3.2. *([11]) (**Pullback attractor**)*
A family of nonempty compact sets $\mathscr{A}=\{\mathcal{A}_\sigma\}_{\sigma\in\Sigma}$ of X is called a pullback attractor for the cocycle φ w.r.t. θ, if for all $\sigma\in\Sigma$, it satisfies

(i) $\varphi(t,\sigma;\mathcal{A}_\sigma)=\mathcal{A}_{\theta_t(\sigma)}$ for all $t\in\mathbb{R}^+$ $\quad$ ($\varphi-$invariance);

(ii) $\lim\limits_{t\to+\infty} dist_X(\varphi(t;\theta_{-t}(\sigma);B),\,\mathcal{A}_\sigma)=0$ for all bounded set $B\subset X$.

Often, $\mathcal{A}_\sigma$ is called a fiber at parameter $\sigma\in\Sigma$.

Definition 3.3. *([11]) Let φ be a cocycle w.r.t. θ on $\mathbb{R}^+\times\Sigma\times X$, and let $B\in\mathcal{B}$. We define the pullback ω-limit set $\omega_\sigma(B)$ as follows*

$$\omega_\sigma(B)=\bigcap_{s\geqslant 0}\overline{\bigcup_{t\geqslant s}\varphi(t,\theta_{-t}(\sigma);B)},\qquad \sigma\in\Sigma,$$

where $\overline{A}$ means the closure of A in X.

If the parameter space Σ contains only one element σ_0 and $\theta_t(\sigma_0)\equiv\sigma_0$ for all $t\in\mathbb{R}$, then φ reduces to a semigroup and all the concepts in *Definitions* 3.1-3.3 coincide with the corresponding concepts in autonomous systems. Especially, in the autonomous case, the pullback attractor coincides with the global attractor; see [3, 35, 36, 40, 24]. Moreover, Chepyzhov & Vishik [14] define the concept of kernel sections for non-autonomous dynamical systems, which correspond to the fibers $\mathcal{A}_\sigma$ in the above Definition 3.2 of a pullback attractor. Furthermore, similar to the autonomous cases, we have also the following equivalent characterization about the pullback ω-limit set.

Lemma 3.4. *([11]) For any $B\subset\mathcal{B}$ and any $\sigma\in\Sigma$, $x_0\in\omega_\sigma(B)$ if and only if there exist $\{x_n\}\subset B$ and $\{t_n\}\subset\mathbb{R}^+$ with $t_n\to+\infty$ as $n\to\infty$, such that*

$$\varphi(t_n,\theta_{-t_n}(\sigma);x_n)\to x_0\quad\textit{as } n\to\infty.$$

Now we define the pullback κ-contracting cocycle in terms of the Kuratowski non-compactness measure:

Definition 3.5. *(κ-**contracting cocycle**)*
Let φ be a cocycle w.r.t. θ on $\mathbb{R}^+ \times \Sigma \times X$. Then φ is called pullback κ-contracting if for any $\varepsilon > 0$, $\sigma \in \Sigma$ and any $B \in \mathcal{B}$, there is a $T = T(\varepsilon, \sigma, B) \geqslant 0$ such that

$$\kappa_X\left(\varphi(t, \theta_{-t}(\sigma); B)\right) \leqslant \varepsilon \quad \text{for all } t \geqslant T.$$

From the definitions above, we have the following basic fact.

Lemma 3.6. *Let φ be a cocycle w.r.t. θ on $\mathbb{R}^+ \times \Sigma \times X$. If φ has a pullback attractor, then φ has a bounded pullback absorbing set and φ is pullback κ-contracting.*

We introduce another definition, needed for characterizations of existence of pullback attractors later.

Definition 3.7. *(**Nested pullback absorbing set**)*
A family of bounded sets $\mathscr{B} = \{B_\sigma\}_{\sigma \in \Sigma}$ of X is called a nested bounded pullback absorbing set for φ w.r.t. θ if $\mathscr{B}$ is a bounded pullback absorbing set, and, moreover, B_σ satisfy the nested relation: $B_{\theta_{-t}(\sigma)} \subset B_\sigma$ for any $t \geqslant 0$ and any $\sigma \in \Sigma$.

Remark 3.8. *This nested relation appears in some systems arising in physical applications. For example, the non-autonomous systems considered in [6, 11, 14] have nested bounded pullback absorbing sets.*

In the following, we will present some characterizations for the pullback κ-contracting cocycles.

Lemma 3.9. *Let φ be a κ-contracting cocycle w.r.t. θ on $\mathbb{R}^+ \times \Sigma \times X$ and have a nested bounded pullback absorbing set $\mathscr{B} = \{B_\sigma\}_{\sigma \in \Sigma}$. Then for every $\sigma \in \Sigma$, every bounded sequence $\{x_n\}_{n=1}^\infty \subset X$ and every time sequence $\{t_n\} \subset \mathbb{R}^+$ with $t_n \to +\infty$ as $n \to \infty$, we have*

(i) $\{\varphi(t_n, \theta_{-t_n}(\sigma); x_n)\}_{n=1}^\infty$ is pre-compact in X;

(ii) all clusters of $\{\varphi(t_n, \theta_{-t_n}(\sigma); x_n)\}_{n=1}^\infty$ are contained in $\omega_\sigma(B_\sigma)$, that is, if

$$\varphi(t_{n_j}, \theta_{-t_{n_j}}(\sigma); x_{n_j}) \to x_0 \quad \text{as } j \to \infty,$$

then $x_0 \in \omega_\sigma(B_\sigma)$;

(iii) $\omega_\sigma(B_\sigma)$ is nonempty and compact in X.

Proof. (i). Denote $\{x_n\}_{n=1}^\infty$ by B. For any $\varepsilon > 0$ and for each $\sigma \in \Sigma$, by the definition of pullback κ-contracting cocycle, we know that there exists a $T_0 = T_0(\varepsilon, \sigma, B_\sigma) > 0$ such that

$$\kappa_X\left(\varphi(t, \theta_{-t}(\sigma); B_\sigma)\right) \leqslant \varepsilon \quad \text{for all } t \geqslant T_0 \tag{3.1}$$

and there exists also a $T_1 = T_1(\varepsilon, \sigma, B)$ such that

$$\varphi(t + T_1, \theta_{-(t+T_1)}(\theta_{-T_0}(\sigma)); B) \subset B_{\theta_{-T_0}(\sigma)} \subset B_\sigma \quad \text{for all } t \geqslant 0. \tag{3.2}$$

Hence, for any $t \geqslant 0$, we have

$$\begin{aligned}
&\varphi(t+T_1+T_0, \theta_{-(t+T_1+T_0)}(\sigma);\, B) \\
&\qquad = \varphi(T_0,\, \theta_{-T_0}(\sigma);\, \varphi(t+T_1,\, \theta_{-(t+T_1)}(\theta_{-T_0}(\sigma)); B)) \\
&\qquad \subset \varphi(T_0,\, \theta_{-T_0}(\sigma);\, B_{\theta_{-T_0}(\sigma)}) \\
&\qquad \subset \varphi(T_0,\, \theta_{-T_0}(\sigma);\, B_\sigma), \qquad (3.3)
\end{aligned}$$

and then

$$\bigcup_{t \geqslant T_0+T_1} \varphi(t, \theta_{-t}(\sigma);\, B) \subset \varphi(T_0,\, \theta_{-T_0}(\sigma);\, B_\sigma). \qquad (3.4)$$

Therefore, combining (3.1) and (3.4), we have

$$\kappa_X \left(\bigcup_{t \geqslant T_0+T_1} \varphi(t, \theta_{-t}(\sigma);\, B) \right) \leqslant \varepsilon. \qquad (3.5)$$

Then by the properties (1), (2) of *Lemma* 2.2 and $\{\varphi(t_n,\, \theta_{-t_n}(\sigma);\, x_n)\}_{n=n_0}^{\infty} \subset \bigcup_{t \geqslant T_0+T_1} \varphi(t, \theta_{-t}(\sigma);\, B)$ for some n_0, we know that $\kappa_X(\{\varphi(t_n,\, \theta_{-t_n}(\sigma);\, x_n)\}_{n=1}^{\infty}) \leqslant \varepsilon$. Hence by the arbitrariness of ε and property (1) of *Lemma* 2.2, we conclude that $\{\varphi(t_n,\, \theta_{-t_n}(\sigma);\, x_n)\}_{n=1}^{\infty}$ is pre-compact in X.

(ii). Let x_0 be a cluster of $\{\varphi(t_n,\, \theta_{-t_n}(\sigma);\, x_n)\}_{n=1}^{\infty}$, we need to show that $x_0 \in \omega_\sigma(B_\sigma)$. Without loss of generality, we assume that $\varphi(t_n,\, \theta_{-t_n}(\sigma);\, x_n) \to x_0$ as $n \to \infty$.

We claim first that for each sequence $\{s_m\}_{m=1}^{\infty} \subset \mathbb{R}^+$ satisfying $s_m \to \infty$ as $m \to \infty$, we can find two sequences $\{t_{n_m}\}_{m=1}^{\infty} \subset \{t_n\}_{n=1}^{\infty}$ and $\{y_m\}_{m=1}^{\infty} \subset B_\sigma$ satisfying $t_{n_m} \to \infty$ as $m \to \infty$, such that

$$\varphi(s_m,\, \theta_{-s_m}(\sigma);\, y_m) = \varphi(t_{n_m},\, \theta_{-t_{n_m}}(\sigma);\, x_{n_m}). \qquad (3.6)$$

Indeed, for each $m \in \mathbb{N}$, we can take n_m so large that $t_{n_m} \geqslant s_m$ and

$$y_m \triangleq \varphi(t_{n_m} - s_m,\, \theta_{-(t_{n_m}-s_m)}(\theta_{-s_m}(\sigma));\, x_{n_m}) \in B_{\theta_{-s_m}(\sigma)} \subset B_\sigma.$$

Therefore,

$$\begin{aligned}
&\varphi(t_{n_m}, \theta_{-t_{n_m}}(\sigma);\, x_{n_m}) \\
&\qquad = \varphi(s_m + (t_{n_m} - s_m),\, \theta_{-(s_m+(t_{n_m}-s_m))}(\sigma);\, x_{n_m}) \\
&\qquad = \varphi(s_m,\, \theta_{-s_m}(\sigma);\, \varphi(t_{n_m} - s_m,\, \theta_{-(t_{n_m}-s_m)}(\theta_{-s_m}(\sigma));\, x_{n_m})) \\
&\qquad = \varphi(s_m,\, \theta_{-s_m}(\sigma);\, y_m). \qquad (3.7)
\end{aligned}$$

Hence,

$$\lim_{m\to\infty} \varphi(s_m,\, \theta_{-s_m}(\sigma);\, y_m) = \lim_{m\to\infty} \varphi(t_{n_m},\, \theta_{-t_{n_m}}(\sigma);\, x_{n_m}) = x_0,$$

and $y_m \in B_\sigma$ for each $m \in \mathbb{N}$, which implies, by the definition of $\omega_\sigma(B_\sigma)$, that $x_0 \in \omega_\sigma(B_\sigma)$.

(iii). The fact that $\omega_\sigma(B_\sigma)$ is nonempty is obvious. Substitute B by B_σ in (3.2)-(3.5), we obtain that there exists a $T_2 = T_2(\varepsilon, B_\sigma, \sigma)$ such that

$$\kappa_X \left(\overline{\bigcup_{t \geqslant T_0+T_2} \varphi(t, \theta_{-t}(\sigma);\, B_\sigma)} \right) = \kappa_X \left(\bigcup_{t \geqslant T_0+T_2} \varphi(t, \theta_{-t}(\sigma);\, B_\sigma) \right) \leqslant \varepsilon.$$

Then by the definition of pullback ω-limit set and property (4) of *Lemma* 2.2, we know that $\omega_\sigma(B_\sigma)$ is compact in X. ■

A criterion for the existence of pullback attractors is then obtained by means of κ-contraction.

Theorem 3.10. ***(Sufficient condition for existence of pullback attractors)*** *Let φ be a continuous cocycle w.r.t. θ on $\mathbb{R}^+ \times \Sigma \times X$. Then (φ, θ) has a pullback attractor provided that*

(i) (φ, θ) *has a nested bounded pullback absorbing set* $\mathscr{B} = \{B_\sigma\}_{\sigma\in\Sigma}$*;*

(ii) (φ, θ) *is pullback κ-contracting.*

Proof. For any $\sigma \in \Sigma$, we consider a family of ω-limit sets $\mathscr{B} = \{B_\sigma\}_{\sigma\in\Sigma}$:

$$\omega_\sigma(B_\sigma) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} \varphi(t, \theta_{-t}(\sigma); B_\sigma)}, \quad \sigma \in \Sigma.$$

By *Lemma* 3.9 we know that $\omega_\sigma(B_\sigma)$ is nonempty and compact in X for each $\sigma \in \Sigma$.

In the following, we will prove that $\mathscr{A} = \{\omega_\sigma(B_\sigma)\}_{\sigma\in\Sigma}$ is a pullback attractor of $(\varphi,\, \theta)$, which will be accomplished in two steps.

Claim 1. For each $\sigma \in \Sigma$ and any $B \in \mathcal{B}$, we have

$$\lim_{t\to+\infty} dist_X(\varphi(t,\, \theta_{-t}(\sigma); B),\, \omega_\sigma(B_\sigma)) = 0.$$

In fact, if *Claim 1* is not true, then there exist $\varepsilon_0 > 0$, $\{x_n\}_{n=1}^\infty \subset B$ and $\{t_n\}$ with $t_n \to +\infty$ as $n \to \infty$, such that

$$dist_X(\varphi(t_n,\, \theta_{-t_n}(\sigma);\, x_n),\, \omega_\sigma(B_\sigma)) \geqslant \varepsilon_0 \quad \text{for } n = 1, 2, \cdots. \tag{3.8}$$

However, thanks to *Lemma* 3.9, we know that $\{\varphi(t_n,\, \theta_{-t_n}(\sigma);\, x_n)\}_{n=1}^\infty$ is pre-compact in X. Without loss of generality, we assume that

$$\varphi(t_n,\, \theta_{-t_n}(\sigma);\, x_n) \to x_0 \quad \text{as } n \to \infty. \tag{3.9}$$

Then $x_0 \in \omega_\sigma(B_\sigma)$, which is a contraction with (3.8). This complete the proof of *Claim 1.*

Claim 2. $\mathscr{A} = \{\omega_\sigma(B_\sigma)\}_{\sigma\in\Sigma}$ *is φ invariant, that is,*

$$\varphi(t,\, \sigma;\, \omega_\sigma(B_\sigma)) = \omega_{\theta_t(\sigma)}(B_{\theta_t(\sigma)}) \quad \textit{for all } t \geqslant 0,\ \sigma \in \Sigma.$$

We first take $x \in \varphi(t,\, \sigma;\, \omega_\sigma(B_\sigma))$.

Then there is a $y \in \omega_\sigma(B_\sigma)$ such that $x = \varphi(t,\, \sigma;\, y)$, and by the definition of y, there exist $\{y_n\} \subset B_\sigma \subset B_{\theta_t(\sigma)}$ and t_n with $t_n \to \infty$ as $n \to \infty$ such that $y = \lim_{n\to\infty} \varphi(t_n,\, \theta_{-t_n}(\sigma);\, y_n)$.

Therefore, by the continuity of φ, as $n \to \infty$,

$$\varphi(t_n + t,\, \theta_{-(t_n+t)}(\theta_t(\sigma));\, y_n) = \varphi(t,\, \sigma;\, \varphi(t_n,\, \theta_{-t_n}(\sigma);\, y_n)) \to \varphi(t,\, \sigma;\, y) = x. \tag{3.10}$$

On the other hand, from *Lemma* 3.9, we know that $\{\varphi(t_n + t,\, \theta_{-(t_n+t)}(\theta_t(\sigma));\, y_n)\}_{n=1}^\infty$ is pre-compact in X. Without loss of generality, we assume that

$$\varphi(t_n + t,\, \theta_{-(t_n+t)}(\theta_t(\sigma));\, y_n) \to x_0 \in \omega_{\theta_t(\sigma)}(B_{\theta_t(\sigma)}) \quad \text{as } n \to \infty.$$

Then by the uniqueness of limitation, we have $x = x_0$, which implies that $x \in \omega_{\theta_t(\sigma)}(B_{\theta_t(\sigma)})$, and thus

$$\varphi(t,\, \sigma;\, \omega_\sigma(B_\sigma)) \subset \omega_{\theta_t(\sigma)}(B_{\theta_t(\sigma)}). \tag{3.11}$$

Now, we only need to prove the converse inclusion relation.

Let $z \in \omega_{\theta_t(\sigma)}(B_{\theta_t(\sigma)})$. Then there exist $\{z_n\} \subset B_{\theta_t(\sigma)}$ and t_n with $t_n \to \infty$ as $n \to \infty$ such that $z = \lim_{n\to\infty} \varphi(t_n,\, \theta_{-t_n}(\theta_t(\sigma));\, z_n)$.

Since $\{z_n\} \subset B_{\theta_t(\sigma)}$ is bounded, from *Lemma* 3.9, we know that $\{\varphi(t_n - t,\, \theta_{-(t_n-t)}(\sigma);\, z_n)\}_{n=1}^{\infty}$ is pre-compact in X. Without loss of generality, we assume that $\varphi(t_n - t,\, \theta_{-(t_n-t)}(\sigma);\, z_n) \to x_0 \in \omega_\sigma(B_\sigma)$ as $n \to \infty$. Then by the continuity of φ, we have

$$\begin{aligned} \varphi(t,\, \sigma;\, x_0) \leftarrow \varphi(t, \sigma;\, &\varphi(t_n - t,\, \theta_{-(t_n-t)}(\sigma);\, z_n)) \\ &= \varphi(t_n,\, \theta_{-(t_n-t)}(\sigma);\, z_n) \\ &= \varphi(t_n,\, \theta_{-(t_n)}(\theta_t(\sigma));\, z_n) \to z. \end{aligned} \tag{3.12}$$

Hence, $z = \varphi(t,\, \sigma;\, x_0)$ with $x_0 \in \omega_\sigma(B_\sigma)$, which implies

$$\omega_{\theta_t(\sigma)}(B_{\theta_t(\sigma)}) \subset \varphi(t,\, \sigma;\, \omega_\sigma(B_\sigma)). \tag{3.13}$$

Combining (3.11) and (3.13) we know that *Claim 2* is true.

From *Claim 1* and *Claim 2*, we complete the proof of *Theorem* 3.10. ■

Remark 3.11. *In the proof of Theorem 3.10, the continuity of the cocycle $\varphi(t,\, \sigma;\, \cdot) : X \to X$ can be replaced by the "weaker" continuity; see* (3.10) *and* (3.12). *That is, the above proof holds for norm-to-weak continuous cocycles; see [41] for autonomous cases.*

Similar to the definition in Caraballo et al [7], we define the following *pullback asymptotic compactness* for NDS.

Definition 3.12. *Let φ be a cocycle w.r.t. θ on $\mathbb{R}^+ \times \Sigma \times X$. Then φ is called pullback asymptotically compact, if for each $\sigma \in \Sigma$, every bounded sequence $\{x_n\}_{n=1}^{\infty}$, and every time sequence $\{t_n\} \subset \mathbb{R}^+$ with $t_n \to +\infty$ as $n \to \infty$, $\{\varphi(t_n,\, \theta_{-t_n}(\sigma);\, x_n)\}_{n=1}^{\infty}$ is pre-compact in X.*

In the framework of pullback attractors, the pullback asymptotic compactness may not be equivalent to the κ-contraction if the cocycle only has a general bounded pullback absorbing set. In fact, we need this bounded pullback absorbing set to satisfy an additional nesting condition; see the next theorem.

From *Lemma* 3.9 we know that if φ has a nested bounded pullback absorbing set, then φ being pullback κ-contracting implies that φ being pullback asymptotically compact; furthermore, in the proof of *Theorem* 3.10, we note that we indeed only used the pullback asymptotic compactness. This, combining with *Lemma* 3.6, implies the following criterion.

Theorem 3.13. ***(Criterion for existence of pullback attractor)***
Let φ be a norm-to-weak continuous cocycle w.r.t. θ on $\mathbb{R}^+ \times \Sigma \times X$ such that (φ, θ) has a nested bounded pullback absorbing set. Then (φ, θ) has a pullback attractor if and only if (φ, θ) is pullback κ-contracting, or equivalently, (φ, θ) is pullback asymptotically compact.

That is, under the assumption that (φ, θ) has a nested bounded pullback absorbing set, pullback κ-contraction is equivalent to pullback asymptotic compactness.

On the other hand, the authors in [7] have proven that (φ, θ) has a pullback attractor provided that (φ, θ) is pullback asymptotically compact and has a bounded pullback absorbing set (see *Theorem* 7 of [7]). In fact, from the definition of pullback attractor, *Lemma* 3.6, Theorems 3.10 and 3.13, we observe that these conditions are also necessary. We summarize this result in the following theorem.

Theorem 3.14. ***(Another criterion for existence of pullback attractor)*** *Let φ be a norm-to-weak continuous cocycle w.r.t. θ on $\mathbb{R}^+ \times \Sigma \times X$. Then (φ, θ) has a pullback attractor if and only if (φ, θ) is pullback asymptotically compact and has a bounded pullback absorbing set.*

$Theorem$ 3.13 and $Theorem$ 3.14 show that pullback asymptotically compact is stronger than pullback $\kappa-$contracting to some extent, which is different from the autonomous cases. Note also that $Theorem$ 3.14 is a slight improvement of $Theorem$ 7 in [7], from continuous cocycles to "weakly" continuous cocycles (i.e., norm-to-weak continuous cocycles).

Although we only use the pullback asymptotic compactness in our later applications in §5, we think that the pullback $\kappa-$contraction criterion for existence of pullback attractors for "weakly" continuous cocycles (i.e., norm-to-weak continuous cocycles) is of independent interest and will be useful for other non-autonomous dynamical systems. Another reason to present the $\kappa-$contraction criterion here is that we like to highlight a difference with the *autonomous* systems: In non-autonomous systems, the pullback asymptotic compactness criterion and the pullback $\kappa-$contraction criterion, for existence of pullback attractors, are not equivalent unless when there exists a *nested* bounded absorbing set ($Theorem$ 3.13).

We also remark that the definitions and results in this section can be expressed in the framework of *processes*, instead of cocycles, as in [14].

4 A technical method for verifying pullback asymptotic compactness

We now present a convenient method for verifying the pullback asymptotic compactness for the cocycle generated by *non-autonomous* hyperbolic type of equations, in order to apply $Theorem$ 3.13 to obtain existence of pullback attractors in the next section. This method is partially motivated by the methods in [17, 18, 25] in some sense; see also in [39]. In [18], the authors present a general abstract framework for asymptotic dynamics of *autonomous* wave equations.

Definition 4.1. *([39]) Let X be a Banach space and B be a bounded subset of X. We call a function $\psi(\cdot,\cdot)$, defined on $X\times X$, a contractive function on $B\times B$ if for any sequence $\{x_n\}_{n=1}^{\infty}\subset B$, there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}\subset\{x_n\}_{n=1}^{\infty}$ such that*

$$\lim_{k\to\infty}\lim_{l\to\infty}\psi(x_{n_k},\,x_{n_l})=0.$$

We denote the set of all contractive functions on $B\times B$ by $Contr(B)$.

Theorem 4.2. ***(Technique for verifying pullback asymptotic compactness)*** *Let φ be a cocycle w.r.t. θ on $\mathbb{R}^+\times\Sigma\times X$ and have a nested bounded pullback absorbing set $\mathscr{B}=\{B_\sigma\}_{\sigma\in\Sigma}$. Moreover, assume that for any $\varepsilon>0$ and each $\sigma\in\Sigma$, there exist $T=T(B_\sigma,\varepsilon)$ and $\psi_{T,\sigma}(\cdot,\cdot)\in Contr(B_\sigma)$ such that*

$$\|\varphi(T,\,\theta_{-T}(\sigma);x)-\varphi(T,\,\theta_{-T}(\sigma);y)\|\leqslant\varepsilon+\psi_{T,\sigma}(x,\,y)\quad \textit{for all } x,y\in B_\sigma,$$

where $\psi_{T,\sigma}$ depends on T and σ. Then φ is pullback asymptotically compact in X.

Proof. Let $\{y_n\}_{n=1}^{\infty}$ be a bounded sequence of X and $\{t_n\}\subset\mathbb{R}^+$ with $t_n\to\infty$ as $n\to\infty$. We need to show that

$$\{\varphi(t_n,\,\theta_{-t_n}(\sigma);\,y_n)\}_{n=1}^{\infty} \text{ is precompact in } X \text{ for each } \sigma\in\Sigma. \tag{4.1}$$

In the following, we will prove that $\{\varphi(t_n,\,\theta_{-t_n}(\sigma);\,y_n)\}_{n=1}^{\infty}$ has a convergent subsequence via diagonal methods (e.g., see [25]).

Taking $\varepsilon_m > 0$ with $\varepsilon_m \to 0$ as $m \to \infty$.

At first, for ε_1, by the assumptions, there exist $T_1 = T_1(\varepsilon_1)$ and $\psi_1(\cdot,\cdot) \in Contr(B_\sigma)$ such that

$$\|\varphi(T_1,\,\theta_{-T_1}(\sigma);\,x) - \varphi(T_1,\,\theta_{-T_1}(\sigma);\,y)\| \leqslant \varepsilon_1 + \psi_1(x,\,y) \quad \text{for all } x, y \in B_\sigma, \tag{4.2}$$

where ψ_1 depends on T_1 and σ.

Since $t_n \to \infty$, for such fixed T_1, without loss of generality, we assume that t_n is so large that

$$\varphi(t_n - T_1,\,\theta_{-(t_n-T_1)}(\theta_{-T_1}(\sigma));\,y_n) \in B_{\theta_{-T_1}(\sigma)} \subset B_\sigma \quad \text{for each } n = 1, 2, \cdots. \tag{4.3}$$

Set $x_n = \varphi(t_n - T_1,\,\theta_{-(t_n-T_1)}(\theta_{-T_1}(\sigma));\,y_n)$. Then from (4.2) we have

$$\begin{aligned}&\|\varphi(t_n,\,\theta_{-t_n}(\sigma);\,y_n) - \varphi(t_m,\,\theta_{-t_m}(\sigma);\,y_m)\| \\ &\qquad = \|\varphi(T_1,\theta_{-T_1}(\sigma);x_n) - \varphi(T_1,\theta_{-T_1}(\sigma);x_m)\| \leqslant \varepsilon_1 + \psi_1(x_n,\,x_m).\end{aligned} \tag{4.4}$$

Due to the definition of $Contr(B_\sigma)$ and $\psi_1(\cdot,\cdot) \in Contr(B_\sigma)$, we know that $\{x_n\}_{n=1}^{\infty}$ has a subsequence $\{x_{n_k}^{(1)}\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty}\lim_{l\to\infty}\psi_1(x_{n_k}^{(1)},\,x_{n_l}^{(1)}) \leqslant \frac{\varepsilon_1}{2}, \tag{4.5}$$

and similar to [25], we have

$$\begin{aligned}&\lim_{k\to\infty}\sup_{p\in\mathbb{N}}\|\varphi(t_{n_{k+p}}^{(1)},\,\theta_{-t_{n_{k+p}}^{(1)}}(\sigma);\,y_{n_{k+p}}^{(1)}) - \varphi(t_{n_k}^{(1)},\,\theta_{-t_{n_k}^{(1)}}(\sigma);\,y_{n_k}^{(1)})\| \\ &\qquad\leqslant \lim_{k\to\infty}\sup_{p\in\mathbb{N}}\limsup_{l\to\infty}\|\varphi(t_{n_{k+p}}^{(1)},\,\theta_{-t_{n_{k+p}}^{(1)}}(\sigma);\,y_{n_{k+p}}^{(1)}) - \varphi(t_{n_l}^{(1)},\,\theta_{-t_{n_l}^{(1)}}(\sigma);\,y_{n_l}^{(1)})\| \\ &\qquad\quad + \limsup_{k\to\infty}\limsup_{l\to\infty}\|\varphi(t_{n_k}^{(1)},\,\theta_{-t_{n_k}^{(1)}}(\sigma);\,y_{n_k}^{(1)}) - \varphi(t_{n_l}^{(1)},\,\theta_{-t_{n_l}^{(1)}}(\sigma);\,y_{n_l}^{(1)})\| \\ &\qquad\leqslant \varepsilon_1 + \lim_{k\to\infty}\sup_{p\in\mathbb{N}}\lim_{l\to\infty}\psi_1(x_{n_{k+p}}^{(1)},\,x_{n_l}^{(1)}) + \varepsilon_1 + \lim_{k\to\infty}\lim_{l\to\infty}\psi_1(x_{n_k}^{(1)},\,x_{n_l}^{(1)}),\end{aligned}$$

which, combining with (4.4) and (4.5), implies that

$$\lim_{k\to\infty}\sup_{p\in\mathbb{N}}\|\varphi(t_{n_{k+p}}^{(1)},\,\theta_{-t_{n_{k+p}}^{(1)}}(\sigma);\,y_{n_{k+p}}^{(1)}) - \varphi(t_{n_k}^{(1)},\,\theta_{-t_{n_k}^{(1)}}(\sigma);\,y_{n_k}^{(1)})\| \leqslant 4\varepsilon_1.$$

Therefore, there is a K_1 such that

$$\|\varphi(t_{n_k}^{(1)},\,\theta_{-t_{n_k}^{(1)}}(\sigma);\,y_{n_k}^{(1)}) - \varphi(t_{n_l}^{(1)},\,\theta_{-t_{n_l}^{(1)}}(\sigma);\,y_{n_l}^{(1)})\| \leqslant 5\varepsilon_1 \quad \text{for all } k, l \geqslant K_1.$$

By induction, we obtain that, for each $m \geqslant 1$, there is a subsequence $\{\varphi(t_{n_k}^{(m+1)},\,\theta_{-t_{n_k}^{(m+1)}}(\sigma);\,y_{n_k}^{(m+1)})\}_{k=1}^{\infty}$ of $\{\varphi(t_{n_k}^{(m)},\,\theta_{-t_{n_k}^{(m)}}(\sigma);\,y_{n_k}^{(m)})\}_{k=1}^{\infty}$ and certain K_{m+1} such that

$$\|\varphi(t_{n_k}^{(m+1)},\,\theta_{-t_{n_k}^{(m+1)}}(\sigma);\,y_{n_k}^{(m+1)}) - \varphi(t_{n_l}^{(m+1)},\,\theta_{-t_{n_l}^{(m+1)}}(\sigma);\,y_{n_l}^{(m+1)})\| \leqslant 5\varepsilon_{m+1} \quad \text{for all } k, l \geqslant K_{m+1}.$$

Now, we consider the diagonal subsequence $\{\varphi(t_{n_k}^{(k)},\,\theta_{-t_{n_k}^{(k)}}(\sigma);\,y_{n_k}^{(k)})\}_{k=1}^{\infty}$. Since for each $m \in \mathbb{N}$, $\{\varphi(t_{n_k}^{(k)},\,\theta_{-t_{n_k}^{(k)}}(\sigma);\,y_{n_k}^{(k)})\}_{k=m}^{\infty}$ is a subsequence of $\{\varphi(t_{n_k}^{(m)},\,\theta_{-t_{n_k}^{(m)}}(\sigma);\,y_{n_k}^{(m)})\}_{k=1}^{\infty}$, then,

$$\|\varphi(t_{n_k}^{(k)},\,\theta_{-t_{n_k}^{(k)}}(\sigma);\,y_{n_k}^{(k)}) - \varphi(t_{n_l}^{(l)},\,\theta_{-t_{n_l}^{(l)}}(\sigma);\,y_{n_l}^{(l)})\| \leqslant 5\varepsilon_m \quad \text{for all } k, l \geqslant \max\{m, K_m\},$$

which, combining with $\varepsilon_m \to 0$ as $m \to \infty$, implies that $\{\varphi(t_{n_k}^{(k)}, \theta_{-t_{n_k}^{(k)}}(\sigma); y_{n_k}^{(k)})\}_{k=1}^{\infty}$ is a Cauchy sequence in X. This shows that $\{\varphi(t_n, \theta_{-t_n}(\sigma); y_n)\}_{n=1}^{\infty}$ is precompact in X for each $\sigma \in \Sigma$. We thus complete the proof. ■

Note that the nested properties and the contractive properties are only used in (4.3) and (4.2) respectively. We have a similar corollary for the cocycle without nested pullback absorbing set, the proof is similar to that for *Theorem* 4.2 above.

Corollary 4.3. *Let φ be a cocycle w.r.t. θ on $\mathbb{R}^+ \times \Sigma \times X$ and have a bounded pullback absorbing set $\mathscr{B} = \{B_\sigma\}_{\sigma\in\Sigma}$. Moreover, assume that for any $\varepsilon > 0$ and each $\sigma \in \Sigma$, there exist $T = T(\sigma, \varepsilon)$ and $\psi_{T,\sigma}(\cdot,\cdot) \in Contr(B_{\theta_{-T}(\sigma)})$ such that*

$$\|\varphi(T, \theta_{-T}(\sigma); x) - \varphi(T, \theta_{-T}(\sigma); y)\| \leqslant \varepsilon + \psi_{T,\sigma}(x, y) \quad \text{for all } x, y \in B_{\theta_{-T}(\sigma)},$$

where $\psi_{T,\sigma}$ depends on T and σ. Then φ is pullback asymptotically compact in X.

5 Pullback attractors for a non-autonomous wave equation

In this section, we prove the existence of the pullback attractor for the non-autonomous wave system (1.1)-(1.3), by applying *Theorems* 3.13 and 3.14. We use the method (via contractive functions) in §4 to verify the pullback asymptotic compactness. This method appears to be very efficient for non-autonomous wave or hyperbolic equations, while the approach in [7], which is an energy method and is different from ours, is very appropriate for some non-autonomous parabolic equations or wave equations with *linear* damping, e.g., see [32]. In fact, the approach in [7] is an energy method and may be seen as a non-autonomous generalization of Ball's method [4].

5.1 Mathematic setting

We consider the non-autonomous wave system (1.1)-(1.3) on the *state space* $X = H_0^1(\Omega) \times L^2(\Omega)$. For each $g_0 \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$, we denote $\{g_0(s+t)|t \in \mathbb{R}\}$ by $\mathcal{H}_1(g_0)$. For $f_0(v, s)$ satisfying (1.8)-(1.9), we similarly denote $\mathcal{H}_2(f_0) = \{f_0(\cdot, s+t)|t \in \mathbb{R}\}$.

Let $\Sigma = \mathcal{H}_2(f_0) \times \mathcal{H}_1(g_0)$ be the parameter space. We define the driving system θ_t: $\Sigma \to \Sigma$ by

$$\theta_t(f_0(\cdot), g_0(\cdot)) = (f_0(t+\cdot), g_0(t+\cdot)), \quad t \in \mathbb{R}. \tag{5.1}$$

Then, system (1.1)-(1.3) is rewritten as the following system

$$\begin{cases} u_{tt} + h(u_t) - \Delta u + f(u, t+s) = g(x, t+s), (x,t) \in \Omega \times \mathbb{R}^+, \\ u(x, t)|_{\partial\Omega} = 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = v_0(x), \end{cases} \tag{5.2}$$

where $s \in \mathbb{R}$ means the initial symbol, corresponding to some $\sigma \in \Sigma$.

Applying monotone operator theory or Faedo-Galerkin method, e.g., see [14, 29, 37], it is known that conditions (1.4)-(1.9) guarantee the existence and uniqueness of strong solution and generalized solution for (1.1)-(1.3), and the time-dependent terms make no essential complications.

Lemma 5.1. ***(Well-posedness)***
Let Ω be a bounded subset of $\mathbb{R}^3$ with smooth boundary, and assume that either Assumption I or Assumption II holds. Then the non-autonomous system (5.2) *has a unique solution* $(u(t),\, u_t(t)) \in \mathcal{C}(\mathbb{R}^+;\, H_0^1(\Omega) \times L^2(\Omega))$ *and* $\partial_t^2 u(t) \in L^2_{loc}(\mathbb{R}^+;\, H^{-1}(\Omega))$ *for any initial data* $x_0 = (u^0,\, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ *and any initial symbol* $\sigma \in \Sigma$.

By *Lemma* 5.1, we can define the cocycle as follows:

$$\begin{cases} \varphi: \quad \mathbb{R}^+ \times \Sigma \times X & \to X, \\ (t, \sigma, (u^0(x), u^1(x))) & \to (u^\sigma(t), u_t^\sigma(t)), \end{cases} \tag{5.3}$$

where $(u^\sigma(t), u_t^\sigma(t))$ is the solution of (1.1) corresponding to initial data $(u^0(x), u^1(x))$ and symbol $\sigma = (f_0(s+\cdot), g_0(s+\cdot))$; and for each $(t,\, \sigma) \in \mathbb{R}^+ \times \Sigma$, the mapping $\varphi(t,\, \sigma;\, \cdot) : X \to X$ is continuous.

Hereafter, we always denote by $(\varphi,\, \theta)$ *the cocycle defined in* (5.1) *and* (5.3).

We now prove the following main result.

Theorem 5.2. ***(Existence of pullback attractor)***
Let Ω be a bounded domain of $\mathbb{R}^3$ with smooth boundary. Then under either Assumption I or Assumption II, the NDS $(\varphi,\, \theta)$ generated by the weak solutions of (1.1)-(1.3) *has a pullback attractor* $\mathscr{A} = \{\omega_\sigma(B_\sigma)\}_{\sigma \in \Sigma}$.

We need a few lemmas before proving this theorem.

5.2 Pullback absorbing sets

In the following, we deal only with the strong solutions of (1.1). The generalized solution case then follows easily by a density argument. We begin with the following existence result on a bounded pullback absorbing set.

Lemma 5.3. ***(Pullback absorbing set)***
Let Ω be a bounded domain of $\mathbb{R}^3$ with smooth boundary. Then under either Assumption I or Assumption II, the NDS $(\varphi,\, \theta)$ has a bounded pullback absorbing set $\mathscr{B} = \{B_\sigma\}_{\sigma \in \Sigma}$.

Proof. For each $\sigma \in \Sigma$, we know that σ is corresponding to some s_0 satisfying that $\sigma = (f(v, s_0 + t), g(x, s_0 + t))$, and $\varphi(t, \theta_{-t_0}(\sigma);\, x_0)$ is the solution of the following equation at time t:

$$\begin{cases} u_{tt} + h(u_t) - \Delta u + f(u, t - t_0 + s_0) = g(x,\, t - t_0 + s_0), \quad (x, t) \in \Omega \times \mathbb{R}^+, \\ (u(0),\, u_t(0)) = x_0, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{5.4}$$

Under *Assumption II*, we can repeat what have done in the proof of [$Theorem$ 1, Haraux[23]], to obtain that there exist a ρ (which depends only on $\|g\|_{L^\infty(\mathbb{R}, L^2(\Omega))}$ and the coefficients in (1.4)-(1.9)) and a T (which depends only on $\|g\|_{L^\infty(\mathbb{R}, L^2(\Omega))}$, the coefficients in (1.4)-(1.9) and the radius of B) such that for any $\sigma \in \Sigma$,

$$\|\varphi(t,\, \theta_{-t_0}(\sigma);\, x_0)\|_X \leqslant \rho \quad \text{for all } T \leqslant t \leqslant t_0 \text{ and } x_0 \in B. \tag{5.5}$$

Hence, for *Assumption II*, we can take $B_\sigma \equiv \{x \in X \mid \|x\|_X \leqslant \rho\}$ for each σ.

Under *Assumption I*, we can use the methods as that in the proof of Chepyzhov and Vishik [[14], *Lemma* 4.1, *Proposition* 4.2, P 121-123], obtain also that there exist C_β and β

(which depend only on the coefficients in (1.4)-(1.9) and C_0) and a T (which depends only on $\int_{-\infty}^{s_0} e^{\beta} \int_\Omega |g(x,s)|^2 dxds$, the coefficients in (1.4)-(1.9) and the radius of B) such that

$$\|\varphi(t, \theta_{-t_0}(\sigma); x_0)\|_X^2 \leqslant \rho_{\sigma,\beta} = C_\beta(1+e^{-\beta s_0} \int_{-\infty}^{s_0} e^{\beta s} \int_\Omega |g(x,s)|^2 dxds) \quad \forall\, T \leqslant t \leqslant t_0,\ x_0 \in B. \tag{5.6}$$

Therefore, under *Assumption I*, we can take $B_\sigma = B_\sigma^\beta = \{x \in X \mid \|x\|_X^2 \leqslant \rho_{\sigma,\beta}\}$. ∎

Remark 5.4. *From* (5.5) *we know that under Assumption II, the NDS* (φ, θ) *has a nested bounded pullback absorbing set* $\mathscr{B} = \{B_\sigma\}_{\sigma\in\Sigma}$.

5.3 Pullback asymptotic compactness

We new prove the pullback asymptotic compactness.

Lemma 5.5. ***(Pullback asymptotic compactness)***
Under either Assumption I or II, for any bounded sequence $\{x_n\}_{n=1}^\infty \in \mathcal{B}$ *and* $\sigma \in \Sigma$*, the sequence* $\varphi(t_n, \theta_{-t_n}(\sigma); x_n\}_{n=1}^\infty$ *is precompact in* X.

The idea for the proof is similar to that in Chueshov & Lasiecka [16, 17, 18] and Khanmamedov [25]; see also in [39] for linear damping and autonomous cases.

In order to prove this lemma on pullback asymptotic compactness, we need to derive a few energy inequalities; see (5.19)-(5.22) below.

We first present some preliminaries and notations.

For each $\sigma \in \Sigma$, we know that σ is corresponding to some s_0 such that $\sigma = (f(v, s_0 + t), g(x, s_0 + t))$. For any $x_0^i = (u_0^i, v_0^i) \in X$ $(i = 1, 2)$, let $(u_i(t), u_{i_t}(t)) = \varphi(t, \theta_{-t_0}(\sigma); x_0^i)$ be the corresponding solution of the following equation at time t:

$$\begin{cases} u_{tt} + h(u_t) - \Delta u + f(u, t - t_0 + s_0) = g(x, t - t_0 + s_0), \quad (x,t) \in \Omega \times \mathbb{R}^+, \\ (u(0), u_t(0)) = x_0^i, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{5.7}$$

For convenience, we introduce notations

$$f_i(t) = f(u_i(t), t - t_0 + s_0), \quad h_i(t) = h(u_{i_t}(t)), \quad t \geqslant 0, \quad i = 1, 2,$$

and

$$w(t) = u_1(t) - u_2(t).$$

Then $w(t)$ satisfies

$$\begin{cases} w_{tt} + h_1(t) - h_2(t) - \Delta w + f_1(t) - f_2(t) = 0, \\ w|_{\partial\Omega} = 0, \\ (w(0), w_t(0)) = (u_0^1, v_0^1) - (u_0^2, v_0^2). \end{cases} \tag{5.8}$$

We also define an energy functional

$$E_w(t) = \frac{1}{2} \int_\Omega |w(t)|^2 + \frac{1}{2} \int_\Omega |\nabla w(t)|^2. \tag{5.9}$$

Since the pullback attractors obtained in *Lemma* 5.3 are different for *Assumption I* and *II*, in the following we will deduce different estimations.

We first deal with the case corresponding to *Assumption II*:

Step 1 Multiplying (5.8) by $w_t(t)$, and integrating over $[s,\, T]\times\Omega$, we obtain

$$E_w(T)+\int_s^T\int_\Omega(h_1(\tau)-h_2(\tau))w_t(\tau)dxd\tau+\int_s^T\int_\Omega(f_1(\tau)-f_2(\tau))w_t(\tau)dxd\tau=E_w(s), \tag{5.10}$$

where $0\leqslant s\leqslant T\leqslant t_0$. Then

$$\int_s^T\int_\Omega(h_1(\tau)-h_2(\tau))w_t(\tau)dxd\tau\leqslant E_w(s)-\int_s^T\int_\Omega(f_1(\tau)-f_2(\tau))w_t(\tau)dxd\tau. \tag{5.11}$$

Combining with *Lemma* 2.3, we get that for any $\delta>0$,

$$\int_s^T\int_\Omega|w_t(\tau)|^2dxd\tau\leqslant|T-s|\delta\cdot mes(\Omega)+C_\delta E_w(s)-C_\delta\int_s^T\int_\Omega(f_1-f_2)w_txd\tau. \tag{5.12}$$

Step 2 Multiplying (5.8) by $w(t)$, and integrating over $[0,\, T]\times\Omega$, we get that

$$\begin{aligned}&\int_0^T\int_\Omega|\nabla w(s)|^2dxds+\int_\Omega w_t(T)\cdot w(T)\\&\quad=\int_0^T\int_\Omega|w_t(s)|^2dxds-\int_0^T\int_\Omega(h_1-h_2)w+\int_\Omega w_t(0)\cdot w(0)-\int_0^T\int_\Omega(f_1-f_2)w.\end{aligned} \tag{5.13}$$

Therefore, from (5.12) and (5.13), we have

$$\begin{aligned}2&\int_0^TE_w(s)ds\\&\leqslant2\delta Tmes(\Omega)+2C_\delta E_w(0)-2C_\delta\int_0^T\int_\Omega(f_1-f_2)w(t)dxds\\&\quad-\int_\Omega w_t(T)w(T)+\int_\Omega w_t(0)w(0)-\int_0^T\int_\Omega(h_1-h_2)w-\int_0^T\int_\Omega(f_1-f_2)w.\end{aligned} \tag{5.14}$$

Integrating (5.10) over $[0,\, T]$ with respect to s, we have that

$$\begin{aligned}TE_w(T)&+\int_0^T\int_s^T\int_\Omega(h_1(\tau)-h_2(\tau))w_t(\tau)dxd\tau ds\\&=-\int_0^T\int_s^T\int_\Omega(f_1-f_2)w_tdxd\tau ds+\int_0^TE_w(s)ds\\&\leqslant-\int_0^T\int_s^T\int_\Omega(f_1-f_2)w_tdxd\tau ds+\delta Tmes(\Omega)+C_\delta E_w(0)\\&\quad-C_\delta\int_0^T\int_\Omega(f_1-f_2)w_tdxds-\frac12\int_\Omega w_t(T)w(T)+\frac12\int_\Omega w_t(0)w(0)\\&\quad-\frac12\int_0^T\int_\Omega(h_1-h_2)w-\frac12\int_0^T\int_\Omega(f_1-f_2)w.\end{aligned} \tag{5.15}$$

Step 3 We will deal with $\int_0^T\int_\Omega(h_1-h_2)w$. Multiplying (5.7) by $u_{i_t}(t)$, we obtain

$$\frac12\frac{d}{dt}\int_\Omega(|u_{i_t}|^2+|\nabla u_i|^2)+\int_\Omega h(u_{i_t})u_{i_t}+\int_\Omega f(u_i,t+s_i)u_{i_t}=\int_\Omega g_iu_{i_t},$$

which, combining with the existence of bounded uniformly absorbing set, implies that

$$\int_0^T \int_\Omega h(u_{i_t})u_{i_t} \leqslant M_T, \tag{5.16}$$

where the constant M_T depends on T (which is different from the autonomous cases). Then, noticing (2.1), we obtain that

$$\int_0^T \int_\Omega |h(u_{i_t})|^{\frac{p+1}{p}} dxds \leqslant M_T. \tag{5.17}$$

Therefore, using H$\ddot{o}$lder inequality, from (5.17) we have

$$|\int_0^T \int_\Omega h_i w| \leqslant M_T^{\frac{p}{p+1}} \left(\int_0^T \int_\Omega |w|^{p+1} \right)^{\frac{1}{p+1}},$$

which implies that

$$|\int_0^T \int_\Omega (h_1 - h_2)w| \leqslant 2M_T^{\frac{p}{p+1}} \left(\int_0^T \int_\Omega |w|^{p+1} \right)^{\frac{1}{p+1}}. \tag{5.18}$$

Hence, combining (5.15) and (5.18), we obtain that

$$\begin{aligned} E_w(T) \leqslant\, & \delta mes(\Omega) - \frac{1}{T}\int_0^T \int_s^T \int_\Omega (f_1(\tau) - f_2(\tau))w_t(\tau)dxd\tau ds + \frac{C_\delta}{T}E_w(0) \\ & - \frac{C_\delta}{T}\int_0^T \int_\Omega (f_1(s) - f_2(s))w_t(s)dxds - \frac{1}{2T}\int_\Omega w_t(T)w(T) + \frac{1}{2T}\int_\Omega w_t(0)w(0) \\ & + \frac{1}{T}M_T^{\frac{p}{p+1}} \left(\int_0^T \int_\Omega |w(s)|^{p+1} dxds \right)^{\frac{1}{p+1}} - \frac{1}{2T}\int_0^T \int_\Omega (f_1(s) - f_2(s))w(s)dxds \end{aligned}$$

for any $0 \leqslant T \leqslant t_0$.

We define

$$\begin{aligned} & \psi_{T,\,\delta,\,\sigma}(x_0^1,\, x_0^2) \\ & = -\frac{1}{T}\int_0^T \int_s^T \int_\Omega (f_1(\tau) - f_2(\tau))w_t(\tau)dxd\tau ds \\ & \quad - \frac{C_\delta}{T}\int_0^T \int_\Omega (f_1(s) - f_2(s))w_t(s)dxds - \frac{1}{2T}\int_\Omega w_t(T)w(T) \\ & \quad + \frac{1}{T}M_T^{\frac{p}{p+1}} \left(\int_0^T \int_\Omega |w(s)|^{p+1} dxds \right)^{\frac{1}{p+1}} - \frac{1}{2T}\int_0^T \int_\Omega (f_1(s) - f_2(s))w(s)dxds. \end{aligned} \tag{5.19}$$

Then we have

$$E_w(T) \leqslant \delta mes(\Omega) + \frac{1}{2T}\int_\Omega w_t(0)w(0) + \frac{C_\delta}{T}E_w(0) + \psi_{T,\,\delta,\,\sigma}(x_0^1,\, x_0^2) \tag{5.20}$$

for any $\delta > 0$, $0 \leqslant T \leqslant t_0$.

For the case corresponding to *Assumption I*:

Since under our general assumption (1.10), as shown in (5.6), the pullback attractors may not satisfy the nested properties. Inspired partly by the results in [7, 26] we will deduce different estimations by the same methods; see (5.21) and (5.22) below.

Repeat *Step 1* and *Step 2* above, and just replace the multipliers $w_t(t)$ and $w(t)$ by $e^{\beta t}w_t(t)$ and $e^{\beta t}w(t)$ respectively, and take into account $\beta < C_0$, we can obtain the following similar estimates

$$E_w(T) \leqslant \frac{\alpha}{TC_0}e^{-\beta T}E_w(0) + \psi'_{T,\,\sigma}(x_0^1,\, x_0^2) \tag{5.21}$$

for any $\delta > 0$, $0 \leqslant T \leqslant t_0$, where $\alpha = (C_0+\beta)/(C_0-\beta)$ and

$$\begin{aligned}
&\psi'_{T,\,\sigma}(x_0^1,\, x_0^2)\\
&= -\frac{e^{-T\beta}}{T}\int_0^T\int_s^T\int_\Omega e^{\beta\tau}(f_1(\tau)-f_2(\tau))w_t(\tau)dxd\tau ds + \frac{\alpha}{2T}e^{-\beta T}\int_\Omega w_t(0)w(0)\\
&\quad - \frac{\alpha}{TC_0}e^{-\beta T}\int_0^T\int_\Omega e^{\beta s}(f_1(s)-f_2(s))w_t(s)dxds - \frac{C}{T}\int_\Omega w_t(T)w(T)\\
&\quad + M_{E_w(0),T,C_0,\beta}\left(\int_0^T\int_\Omega |w(s)|^2dxds\right)^{\frac{1}{2}}.
\end{aligned} \tag{5.22}$$

With the above energy inequalities, we are now ready to prove pullback asymptotic compactness.

Proof of *Lemma* **5.5:** We will deal with *Assumption I* and *Assumption II* separately.

Assumption II:

For each $\sigma \in \Sigma$, and for any fixed $\varepsilon > 0$, from (5.19), we can take t_0 large enough such that

$$E_w(t_0) \leqslant \varepsilon + \psi_{t_0,\,\delta,\,\sigma}(x_0^1,\, x_0^2) \text{ for all } x_0^1,\, x_0^2 \in B_\sigma. \tag{5.23}$$

Hence, thanks to *Theorem* 4.2 and *Lemma* 5.3, it is sufficiently to prove that the function $\psi_{t_0,\,\delta,\,\sigma}(\cdot,\,\cdot)$ defined in (5.19) belongs to $Contr(B_\sigma)$ for each fixed t_0.

We observe from equation (5.7) (and also see [22]) that for any $t_0 > 0$,

$$\bigcup_{t\in[0,\,t_0]} \varphi(t,\,\theta_{-t_0}(\sigma); B_\sigma) \ \text{ is bounded in } X, \tag{5.24}$$

and the bound depends only on t_0 and σ.

Let (u_n, u_{t_n}) be the corresponding solution of $(u_0^n, v_0^n) \in B_\sigma$ for problem (5.7), $n = 1, 2, \cdots$. From the observation above, without loss of generality (or by passing to subsequences), we assume that

$$u_n \to u \quad \star-\text{weakly in } L^\infty(0, t_0;\, H_0^1(\Omega)), \tag{5.25}$$

$$u_n \to u \quad \text{in } L^{p+1}(0, t_0;\, L^{p+1}(\Omega)), \tag{5.26}$$

$$u_{n_t} \to u_t \quad \star-\text{weakly in } L^\infty(0, t_0;\, L^2(\Omega)), \tag{5.27}$$

$$u_n \to u \quad \text{in } L^2(0, t_0;\, L^2(\Omega)) \tag{5.28}$$

and

$$u_n(0) \to u(0) \ \text{ and } \ u_n(t_0) \to u(t_0) \quad \text{in } L^4(\Omega). \tag{5.29}$$

Here we have used the compact embeddings $H_0^1 \hookrightarrow L^4$ and $H_0^1 \hookrightarrow L^{p+1}$ (since $1 \le p < 5$).

Now, we will deal with each term in (5.19) one by one.

First, from (5.24), (5.29) and (5.26) we get that

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_\Omega (u_{n_t}(t_0)-u_{m_t}(t_0))(u_n(t_0)-u_m(t_0))dx = 0, \tag{5.30}$$

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_0^{t_0}\int_\Omega |u_n(s)-u_m(s)|^{p+1}dxds = 0, \tag{5.31}$$

and from (1.8) and (5.28), we further have

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_0^{t_0}\int_\Omega (f(u_n(s),s-t_0+s_0)-f(u_m(s),s-t_0+s_0))(u_n(s)-u_m(s))dxds = 0. \tag{5.32}$$

Second, note that

$$\begin{aligned}
&\int_0^{t_0}\int_\Omega (u_{n_t}(s)-u_{m_t}(s))(f(u_n(s),s-t_0+s_0)-f(u_m(s),s-t_0+s_0))dxds\\
&\quad= \int_0^{t_0}\int_\Omega u_{n_t}(s)f(u_n(s),s-t_0+s_0) + \int_0^{t_0}\int_\Omega u_{m_t}(s)f(u_m(s),s-t_0+s_0)\\
&\qquad - \int_0^{t_0}\int_\Omega u_{n_t}(s)f(u_m(s),s-t_0+s_0) - \int_0^{t_0}\int_\Omega u_{m_t}(s)f(u_n(s),s-t_0+s_0)\\
&\quad= \int_\Omega F(u_n(t_0),s_0) - \int_\Omega F(u_n(0),-t_0+s_0) - \int_0^{t_0}\int_\Omega F_s(u_n(\tau),\tau-t_0+s_0)dxd\tau\\
&\qquad + \int_\Omega F(u_m(t_0),s_0) - \int_\Omega F(u_m(0),-t_0+s_0) - \int_0^{t_0}\int_\Omega F_s(u_m(\tau),\tau-t_0+s_0)dxd\tau\\
&\qquad - \int_0^{t_0}\int_\Omega u_{n_t}(s)f(u_m(s),s-t_0+s_0) - \int_0^{t_0}\int_\Omega u_{m_t}(s)f(u_n(s)s-t_0+s_0).
\end{aligned}$$

By (5.25), (5.27), (5.29) and (1.8), taking first $m\to\infty$ and then $n\to\infty$, we obtain that

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\int_0^{t_0}\int_\Omega (u_{n_t}(s)-u_{m_t}(s))(f(u_n(s),s-t_0+s_0)-f(u_m(s),s-t_0+s_0))dxds\\
&\quad= \int_\Omega F(u(t_0),s_0) - \int_\Omega F(u(0),-t_0+s_0) - \int_0^{t_0}\int_\Omega F_s(u(\tau),\tau-t_0+s_0)dxd\tau\\
&\qquad + \int_\Omega F(u(t_0),s_0) - \int_\Omega F(u(0),-t_0+s_0) - \int_0^{t_0}\int_\Omega F_s(u(\tau),\tau-t_0+s_0)dxd\tau\\
&\qquad - \int_0^{t_0}\int_\Omega u_t f(u(s),s-t_0+s_0) - \int_0^{t_0}\int_\Omega u_t f(u(s),s-t_0+s_0)\\
&\quad= 0.
\end{aligned} \tag{5.33}$$

Similarly, we have

$$\begin{aligned}
&\int_s^{t_0}\int_\Omega (u_{n_t}(\tau)-u_{m_t}(\tau))(f(u_n(\tau),\tau-t_0+s_0)-f(u_m(\tau),\tau-t_0+s_0))dxd\tau\\
&\qquad=\int_\Omega F(u_n(t_0),s_0)-\int_\Omega F(u_n(s),s-t_0+s_0)-\int_s^{t_0}\int_\Omega F_s(u_n(\tau),\tau-t_0+s_0)dxd\tau\\
&\qquad\quad+\int_\Omega F(u_m(t_0),s_0)-\int_\Omega F(u_m(s),s-t_0+s_0)-\int_s^{t_0}\int_\Omega F_s(u_m(\tau),\tau-t_0+s_0)dxd\tau\\
&\qquad\quad-\int_s^{t_0}\int_\Omega u_{n_t} f(u_m(\tau),\tau-t_0+s_0)-\int_s^{t_0}\int_\Omega u_{m_t} f(u_n(\tau),\tau-t_0+s_0).
\end{aligned}$$

Since $|\int_s^{t_0}\int_\Omega (u_{n_t}(\tau)-u_{m_t}(\tau))(f(u_n(\tau),\tau-t_0+s_0)-f(u_m(\tau),\tau-t_0+s_0))dxd\tau|$ is bounded for each fixed t_0, by the Lebesgue dominated convergence theorem, we finally have

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\int_0^{t_0}\int_s^{t_0}\int_\Omega (u_{n_t}(\tau)-u_{m_t}(\tau))(f(u_n(\tau),\tau-t_0+s_0)-f(u_m(\tau),\tau-t_0+s_0))dxd\tau ds\\
&\quad=\int_0^{t_0}\Big(\lim_{m\to\infty}\lim_{n\to\infty}\int_s^{t_0}\int_\Omega (u_{n_t}(\tau)-u_{n_t}(\tau))\\
&\qquad\qquad\qquad(f(u_n(\tau),\tau-t_0+s_0)-f(u_m(\tau),\tau-t_0+s_0))dxd\tau\Big)ds\\
&\quad=\int_0^{t_0} 0ds=0.
\end{aligned}\tag{5.34}$$

Hence, from (5.30)-(5.34), we see that $\psi_{t_0,\delta,\sigma}(\cdot,\,\cdot)\in Contr(B_\sigma)$.

Assumption I:

From the definition of $\rho_{\sigma,\beta}$ (see (5.6)), we have the conclusion: *for each σ and for any $\varepsilon>0$, we can take t_0 large enough such that $e^{-\beta t_0}\rho_{\theta_{-t_0}(\sigma),\beta}\leqslant\varepsilon$.*

Hence, from *Corollary* 4.3 and (5.21), we only need to verify that the function $\psi'_{t_0,\,\sigma}(\cdot,\,\cdot)$ defined in (5.22) belongs to $Contr(B_{\theta_{-t_0}(\sigma)})$. To this end, we notice that $e^{\beta t}$ is bounded in $[0,t_0]$ and $\bigcup_{t\in[0,\,t_0]}\varphi(t,\,\theta_{-t_0}(\sigma);B_{\theta_{-t_0}(\sigma)})$ is bounded in X. The remainder is just a repeat of that for $\psi_{t_0,\,\delta,\,\sigma}(\cdot,\,\cdot)$ above.

This completes the proof of *Lemma* 5.5. ■

5.4 Existence of pullback attractors

Now we complete the proof of the main result.

Proof of *Theorem* 5.2 From *Lemma* 5.3 and *Lemma* 5.5, we see that the conditions of *Theorems* 3.13 and 3.14 are all satisfied respectively and thus we imply the existence of the pullback attractor. ■

Remark 5.6. *In this section, we obtain the pullback asymptotic compactness for the non-autonomous wave system* (1.1)-(1.3) *by the technique presented in §4. This technique is different from the method in [7]. Due to the existence of nested bounded pullback absorbing set for* Assumption II*(Lemma 5.3), the pullback $\kappa-$contraction is equivalent to pullback asymptotic compactness (see Theorem 3.13). Thus, in principle, we could also use the pullback $\kappa-$contraction criterion to conclude the existence of pullback attractor, using the decomposition method as in [19, 38] (popular for autonomous systems).*

6 Some remarks

In this paper, we discuss the asymptotic behavior of solutions in the framework of pullback attractors. Another interesting question is forward attractors; see [11, 13] for general discussions or [6] for practical applications to wave equations with delays. However, for the forward attraction property to hold, one usually needs some uniformity about the time-dependent terms (i.e., about the symbol spaces [14]). As discussed in details in [8, 26], for general non-autonomous dissipative systems, how to obtain the forward attraction properties is an open problem if without this uniformity assumption.

For our problem, under the **Assumption II** in §1, we indeed obtain a bounded uniformly absorbing set in the sense of [14] in *Lemma* 5.3 (or see Haraux[22]). If we assume further that g satisfies some additional conditions, e.g., g is translation compact or $g \in W^{1,\infty}(\mathbb{R}; L^2(\Omega))$, then by the same method, we can verify the family of processes (see [14] for more details) corresponding to the non-autonomous wave system (1.1)-(1.3) is uniformly asymptotically compact and thus has a uniform (w.r.t. $\sigma \in \Sigma$) attractor in the sense of [14]. However, for the case of **Assumption I** in §1, it appears difficult to discuss the forward attraction for g satisfying only (1.10).

For the *autonomous* case of (1.1)-(1.3), recently, Chueshov & Lasiecka [18] have shown a general result for the existence of global attractor, and they allow $p = 5$, i.e., the so-called critical interior damping. In their *autonomous* case, it is true that for all $0 \leqslant s \leqslant t$,

$$\int_s^t \int_\Omega h(u_t) u_t dx d\tau \leqslant C_R, \tag{6.1}$$

where C_R depends only on the norm of initial data, but independent of time instants s and t. However, for our non-autonomous case, this constant may depend on time instants s and t (e.g., see (5.16),(5.17)), and thus in our proofs, we require (at least, technically) that the growth order of h to be strictly less than 5: $p < 5$.

Moreover, in the present paper, we use the *pullback asymptotic compactness* to obtain the existence of pullback attractors of non-autonomous hyperbolic systems. This is mainly based on a technical method for verifying *pullback asymptotic compactness* in §4. However, for other non-autonomous systems or using other techniques (e.g., the decomposition method), the pullback $\kappa-$contraction criterion may be more appropriate for proving the existence of pullback attractors.

Finally, we point out that all the contents in this paper can be expressed by the framework of *processes*, instead of cocycles, as in [6, 9, 8, 26]; see also [14] for more results about processes.

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